A State-Dependent Model of Intermediate Goods Pricing

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Abstract

Recent analyses of transaction-level data sets have generated new stylized facts on price setting and greatly influenced the empirical open- and closed-economy macroeconomics literatures. This work has uncovered marked heterogeneity in price stickiness, demonstrated that even non-zero price changes do not fully "pass through" exchange rate shocks, and offered evidence of synchronization in the timing of price changes. Further, intrafirm prices have been shown to differ from arm's length prices in each of these characteristics. This paper develops a state-dependent model of price setting by strategic intermediate goods producers that anticipate and respond to their competitors' actions. The model, which allows for both arm's length and intrafirm transactions, is able to generate all of these empirical pricing patterns.

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1 Introduction

Recent analyses of transaction-level data sets have generated new stylized facts on price setting and greatly influenced the empirical open- and closed-economy macroeconomics literatures. This work has uncovered marked heterogeneity in price stickiness, demonstrated that even non-zero price changes do not fully "pass through" exchange rate shocks, and offered evidence of synchronization in the timing of price changes. For instance, Bils and Klenow (2004), Nakamura and Steinsson (2008), and Gopinath and Rigobon (2008) demonstrate in retail and trade data that prices of goods with higher elasticities of demand change more frequently. Further, Gopinath, Itskhoki, and Rigobon (2010), Burstein and Jaimovich (2009), and Fitzgerald and Haller (2010) show that even non-zero price changes do not fully "pass through" exchange rate shocks. Finally, Cavallo (2011) and Midrigan (2011) document synchronization in the timing of price changes. This paper develops a state-dependent model of intermediate goods pricing that is capable of simultaneously generating all of these empirical pricing patterns.

Three characteristics of the model are crucial for its ability to match patterns of price stickiness, exchange rate pass-through, and price synchronization. First, the model is state-dependent. Firms have discretion over when to change prices and may opt for price spells of heterogenous lengths even for a given good. Time dependent models, by contrast, exogenously set the degree of price stickiness for each good as a parameter.

Second, each firm in the model faces an elasticity of demand that changes over time. This leads firms to price at a variable markup over marginal cost and to incompletely pass permanent cost shocks through to prices even at the time of a price change. A constant elasticity of substitution (CES) demand system would instead imply complete passthrough of permanent cost shocks at the time of a price change.[1]

Third, the dynamic pricing problem is treated as a game between a finite number of large firms. Each firm is strategic in that it actively considers and responds to each other firm's pricing policy. This is the mechanism through which the model generates synchronization in price changes even with idiosyncratic shocks: a firm might change its price solely because it

[1] A CES demand system can lead to incomplete passthrough at the time of a price change if adjustment is costly and the marginal cost shock is not permanent (mean reverting, for example). Motivated by the interpretation of these shocks as exchange rate movements, we instead focus on the case of permanent (or highly persistent) shocks.

expects its competitor to do so.

An alternative environment with a continuum of atomistic firms that set prices as variable markups over marginal cost could also generate heterogenous stickiness, incomplete passthrough of permanent cost shocks at the time of a price change, and synchronized price changes. Models with continua of firms are far less useful in the presence of large idiosyncratic shocks, however, because atomistic firms are typically modeled as only using aggregate information to forecast relevant state variables.[2] Such setups are ill-equipped to model a sector in which aggregate outcomes depend on the full distribution of firm-level variables. By contrast, the price-setting game described below can be used to generate and study pricing dynamics even in industries that are characterized by large oligopolistic firms and important firm-specific shocks.[3]

Table 1 summarizes how each of the above elements expand on the ability of the simplest models to match the observed patterns in stickiness, passthrough, and price synchronization. A standard time-dependent framework with CES demand and monopolistic competition would be categorized in column (i), unable to produce any of the three underlined empirical patterns listed in the table's rows. Most of the papers cited above include models which either fit cleanly in the class of state-dependent models with constant markups, column (ii), or in the class of state-dependent models with variable markups and atomistic firms, column (iii). This paper presents a model belonging to column (iv), able to simultaneously match all three patterns even in an environment with important idiosyncratic shocks.[4]

I consider a two-firm game in a partial equilibrium environment with upstream manufacturers exporting intermediate inputs to downstream firms. Each upstream manufacturer faces a production cost shock and decides whether to keep its existing price or to pay an adjustment cost to change it. The model allows for the production cost shocks to be correlated, but I focus on the calibration with only idiosyncratic shocks in order to isolate the role of gaming

[2]Gopinath and Itskhoki (2010) and Nakamura and Steinsson (2010), for example, use the methodology of Krusell and Smith (1998) to generate firms' expectations of inflation.

[3]International trade data suggest the prevalence of these types of industries in the traded sector. For example, Bernard et al. (2009) and Gopinath and Neiman (2011) document the high concentration of trade flows among a small number of large firms in U.S. and Argentine trade data.

[4]There are only three other papers I am aware of which focus on pricing dynamics such as stickiness and passthrough and that also fall in column (iv). Nakamura and Zerom (2010) estimate a model of the coffee industry to disaggregate the sources of coffee-price passthrough. Kano (2010) studies the impact of strategic interactions on estimates of menu costs in weekly supermarket scanner data. Slade (1999) estimates a pricing game with data on sales of saltine crackers and determines that strategic interactions increase stickiness.

behavior in synchronizing the timing of price changes. This maps well to the case in which the manufacturers are located in different countries and the cost shocks are largely driven by different exchange rate fluctuations.

The degree to which the cost shock renders the firm's current price suboptimal depends on the firm's elasticity of demand. This elasticity differs across sectors, leading to heterogenous stickiness. Further, the traded inputs are substitutes so a price change by one firm will impact its competitor's elasticity of demand. Each firm's elasticity of demand will therefore fluctuate over time, generating variable markups and incomplete passthrough. Finally, because each firm's actions impact their competitors' profitability, they may induce a response, resulting in synchronization in the timing of price changes.

In addition to matching these three key empirical patterns, the model can also be applied to examine pricing dynamics of intrafirm trades between related parties, a category representing nearly 40 percent of all U.S. imports. Intrafirm trade includes both cross-border trading by parents and subsidiaries of the same multinational corporation and domestic transactions between different business units of the same firm. Bernard, Jensen, and Schott (2006), Hellerstein and Villas-Boas (2010), and Neiman (2010) use micro-data to examine differences in international intrafirm transactions. All three papers find that intrafirm prices exhibit higher exchange rate passthrough than arm's length prices, and Neiman (2010) additionally finds that they exhibit less stickiness and synchronization.[5] I extend the baseline model to allow for intrafirm trade and demonstrate that it is also able to match these patterns.[6]

The model focuses on intermediate good prices, as opposed to final good prices, for three reasons. First, international trade is predominantly in intermediates and most of the empirical regularities discussed above were identified in trade prices. Second, the model is ill-equipped to think about several well-documented characteristics of final good prices including seasonality and frequent sales. Third, the extension of the model to the case of related parties is most natural in a context of intermediate goods. As discussed below, however, most results can be

[5]The results in Neiman (2010) are for the set of differentiated traded goods.

[6]Some readers may reasonably be skeptical of the model's assumption that intrafirm prices are allocative and not mere accounting constructs set, for example, to minimize tax exposure. If the transfer prices observed in government data are not allocative, then the ability of the model to match patterns in that data is, of course, not interesting. Companies may use allocative prices even if they do not report them, though. As such, even if one disregards any comparison to actual data, the model can still yield insight into the differential impact and transmission of shocks in environments with vertical integration.

easily interpreted in terms of final good prices.[7] A model simultaneously generating realistic intermediate and final good pricing dynamics would be highly useful, but is beyond the scope of this paper and left for future research.

In sum, many of the new facts on import, export, producer, and retail prices suggest the need for a dynamic model of price adjustment with at least three features: state-dependent pricing, variable elasticities of demand, and gaming behavior by finite market-share firms. Further, a model which can be easily extended to allow for intrafirm trade is needed to compare pricing characteristics across different vertical production structures. I now describe a partial equilibrium model with these features that is capable of producing the salient facts on arm's length price setting – and the comparison along these dimensions with intrafirm price setting – found across this large set of empirical studies.

2 A Partial Equilibrium Model of Trade in Intermediate Goods

The model is a nested CES structure closely related to that used in Yang (1997) and more recently in Atkeson and Burstein (2008). An infinitely lived representative consumer buys a continuum of final goods that are assembled by distributors from two inputs purchased at arm's length from upstream manufacturers. The cost of production for these inputs varies over time due to idiosyncratic cost shocks. Distributor pricing is completely flexible, while manufacturers must pay a fixed adjustment cost to change their prices.[8] Consumers maximize their lifetime expected utility and arm's length manufacturers maximize their lifetime expected profits.

2.1 Consumers

Consumers maximize their expected lifetime utility from consumption streams at times t, $E_t \sum_{t=0}^{\infty} \beta^t U(C_t)$, where they have a discount factor β and exhibit a CES love of variety over a continuum of final goods c that are indexed by $z \in [0,1]$, yielding $C_t = \left[\int_0^1 c_t(z)^{\frac{\eta-1}{\eta}} dz\right]^{\frac{\eta}{\eta-1}}$.

[7] The pricing problem of the intermediate good producer in the model, where the ultimate customer has CES preferences, is isomorphic to the pricing problem of a stand-alone final good producer facing a nested CES demand.

[8] The assumption of greater flexibility in downstream prices is supported in the data. See, for instance, Shoenle (2010) and Gopinath and Rigobon (2008).

As is standard in this setup, consumer demand for good $c(z)$ is $c_t(z) = C_t \left(p_t(z)\right)^{-\eta} \left(P_t\right)^{\eta}$, where the price index is defined as: $P_t = \left[\int_0^1 p_t \left(z\right)^{1-\eta} dz\right]^{\frac{1}{1-\eta}}$.

2.2 Distributors

There is a continuum of distributors that costlessly assemble each final good using a CES production technology that combines two product-specific manufactured intermediate inputs:

$$c_t \left(z\right) = \left[\gamma(z) c_{1,t}(z)^{\frac{\rho(z)-1}{\rho(z)}} + \left(1-\gamma(z)\right) c_{2,t}(z)^{\frac{\rho(z)-1}{\rho(z)}}\right]^{\frac{\rho(z)}{\rho(z)-1}},$$

where $\eta < \rho(z) < \infty$ and $\gamma(z) \in (0,1)$ for all z. Sectors with higher values of ρ are less differentiated as the distributor can more easily substitute away from any given input in those sectors. Distributors take input prices as given and solve the problem:

$$\max_{p_t(z)} \; p_t \left(z\right) c_t(z) - p_{1,t}(z) c_{1,t}(z) - p_{2,t}(z) c_{2,t}(z), \tag{1}$$

which results in demand for the first manufactured input (expression for the second input, not shown, is symmetric) of:

$$c_{1,t}(z) = c_t \left(z\right) \left(p_{1,t}(z)\right)^{-\rho(z)} \left(\gamma \left(z\right) x_t \left(z\right)\right)^{\rho(z)},$$

where

$$x_t(z) = \left[\gamma(z)^{\rho(z)} p_{1,t} \left(z\right)^{1-\rho(z)} + \left(1-\gamma(z)\right)^{\rho(z)} p_{2,t} \left(z\right)^{1-\rho(z)}\right]^{\frac{1}{1-\rho(z)}}$$

is the total unit production cost of the final good. Distributors then set price at a constant markup over this marginal cost of production, $p_t \left(z\right) = \left(\eta / \left(\eta - 1\right)\right) x_t(z)$.

2.3 Manufacturers

Intermediate good manufacturers use a linear technology to produce $c_{j,t}(z)$ at a constant marginal cost for each firm j, which I write in logs for convenience of notation as: $\ln[m_{j,t}(z)] = \textit{constant} + \lambda e_{j,t}(z)$. $e_{j,t}(z)$ shifts the marginal costs of firm j supplying inputs for final good z at time t. In an open-economy setting, it can alternatively be thought of as an exchange rate.

In a closed-economy setting, it can be thought of as an idiosyncratic productivity term. I focus on the case where the two firms' shocks are uncorrelated (i.e. all shocks are idiosyncratic). I do this to emphasize that the model generates synchronized price changes even without any common cost shock. A model where atomistic firms only use information on common shocks, for example, would not be well-suited to this case. The framework, however, can handle any correlation structure.

A share of the total production costs, $(1-\lambda)$, is impacted by this shock. This captures the case when productivity gains only impact certain production processes, or in an open-economy, when the exchange rate does not fully impact the unit cost because the manufacturer itself imports intermediate inputs from abroad. Though I do not focus on any particular set of quantitative estimates, I introduce λ to come closer to matching the highly incomplete rates of exchange rate passthrough seen in the empirical literature. I model the shock process as an AR(1):

$$e_{j,t}(z) = \delta e_{j,t-1}(z) + \mu_{j,t}(z),$$

where $\mu_{j,t}(z)$ is normally distributed with cumulative distribution function $F_j(\mu_j(z))$. This allows for shocks that are strongly mean-reverting ($\delta < 1$) as well as those arbitrarily close to fully persistent (as $\delta \to 1$).

2.4 Manufacturer Price Setting

Unlike the distributors, the manufacturers pay a fixed cost to change their nominal prices. These trades are business-to-business transactions, and hence, this fixed cost is more typically thought to reflect the cost of changing processes, communicating, and negotiating with customers than the retail price interpretation as "menu" costs (See Zbaracki et al., 2004).

I follow Dotsey, King, and Wolman (1999) in modeling the fixed adjustment cost, $\phi_{j,t}$, as a random variable drawn identically and independently each period from the distributions $G_j(\cdot)$. Firms know the distribution of their competitor's adjustment cost, $G_{-j}(\cdot)$, but they only observe their own realized cost. This assumption of random and private adjustment costs is helpful because it rules out certain cases in which there would be multiple (or no) equilibria. It allows for a game in pure strategies, but where each player treats the other as if she were playing a mixed strategy due to uncertainty about the other's state. As implemented, this

assumption does not impact any of the qualitative results.

Each period, the manufacturer that provides the first input (the setup is symmetric, so I focus on this manufacturer without loss of generality) earns operating profits $\pi_1 = p_1 c_1 - m_1 c_1$, which are defined to exclude the cost of price adjustment. For notational convenience, I drop the sector and time indices, z and t, when they are not needed, and re-write operating profits:

$$\pi_1 = CP^{\eta}\left(\frac{\eta}{\eta-1}\right)^{-\eta}\left[\gamma^{\rho}{p_1}^{1-\rho} + (1-\gamma)^{\rho}{p_2}^{1-\rho}\right]^{\frac{\rho-\eta}{1-\rho}}\gamma^{\rho}{p_1}^{-\rho}\left(p_1 - m_1\right).$$

Manufacturers maximize the present value of real profits, less real adjustment costs $\phi_{j,t}/P_t$, by solving:

$$\max_{p_j(s_t)} \; E_t \sum_{t=0}^{\infty} \beta^t \left[\frac{\pi_{j,t}}{P_t} - \frac{\phi_{j,t}}{P_t}\chi_{j,t}\right]. \tag{2}$$

$\chi_{j,t}$ is an indicator function equalling 1 when $p_{j,t} \neq p_{j,t-1}$ and 0 otherwise. Again, I assume that the menu cost $\phi_{j,t}$ is known at time t.

3 Determinants of Pricing Patterns

In this section, before proceeding to the full dynamic model's solution and simulation, I try to build intuition for the model's ability to match characteristics in the data. I start with the case without nominal rigidities ($\phi_j = 0$). I next add an adjustment cost, take an approximation of the firm's profit function, and run some simple one-period numerical examples. These are designed to generate intuition for the determinants of price duration. These exercises suggest the model will produce the patterns on duration, passthrough, and synchronization found in the empirical literature.

3.1 Flexible Prices

Firms set optimal prices by taking their competitor's price as given and pricing at a variable markup over marginal cost, $p_j = \frac{\varepsilon(s_j)}{\varepsilon(s_j)-1} m_j$. The market share of input manufacturer j in that sector, s_j, can be expressed as:

$$s_j = \frac{p_j c_j}{xc} = \left(1 + \left(\gamma_j/\gamma_{-j}\right)^{-\rho}\left(p_j/p_{-j}\right)^{\rho-1}\right)^{-1},$$

and the elasticity of demand, ε_j, is the market-share weighted average of the elasticities of substitution for final goods and for the sector's intermediate inputs:

$$\varepsilon_j(s_j) = \eta s_j + \rho(1 - s_j).$$

The optimal price depends on both the firm's own cost and, through its impact on market share, the competitor's price. This strategic complementarity is often assumed away in setups with monopolistic competition. Markups decrease with the elasticity of demand, and firms with given market shares will charge lower markups for more substitutable goods.[9]

Totally differentiating the markup, elasticity, and market share definitions, I approximate the change in price as a weighted average of the shocks to a firm's own cost and its competitor's price:

$$\widehat{p_j} = \alpha_j \widehat{m_j} + (1 - \alpha_j)\widehat{p_{-j}} = \lambda \alpha_j \mu_j + (1 - \alpha_j)\widehat{p_{-j}}, \tag{3}$$

where:

$$\alpha_j = \frac{\varepsilon_j(\varepsilon_j - 1)}{\varepsilon_j(\varepsilon_j - 1) + (\rho - \eta)(\rho - 1)s_j(1 - s_j)}, \tag{4}$$

and where $\widehat{x} = dx/x$ denotes the change of a variable x (in logs). Expression (3) measures the responsiveness of the flexible price to a change in marginal cost or to its competitor's price. It assumes that the competitor does not subsequently change its price further. $\lambda\alpha_j$ approximates passthrough of the cost shock for arm's length firms. Given that $\lambda, \alpha_j \in (0, 1)$, passthrough will be incomplete, even after a price changes, consistent with the data. As the elasticity of demand ε_j changes with market share, the markup $\varepsilon_j/(\varepsilon_j - 1)$ will change, and a varying amount of the cost shock will be absorbed, rather than passed through.

Equation (3) makes clear that a change in price by one firm can induce a change in price by the other, leading to price synchronization in the full dynamic model. Substituting $\widehat{p_{-j}} = \alpha_{-j}\widehat{m_{-j}} + (1 - \alpha_{-j})\widehat{p_j}$ into (3), I can write:

$$\widehat{p_j} = \zeta_j \widehat{m_j} + \left(1 - \zeta_j\right)\widehat{m_{-j}},$$

[9]This expression for the intermediate good producer's elasticity of demand, the key determinant of pricing dynamics in the model, is identical to that of a firm producing a final good and facing a consumer with nested CES preferences. Such a consumer would substitute between goods in the same sector with a constant elasticity of ρ and across sectors with an elasticity of η.

where j and $-j$ are the two competing firms, and:

$$\zeta_j = \frac{\alpha_j}{\alpha_j + \alpha_{-j} - \alpha_j \alpha_{-j}} \in (0, 1)$$

is now the equivalent expression to (4) but for the case where the firms fully respond to each other. $\lambda\zeta_j$ would then be the corresponding approximation to cost passthrough. Note that $\zeta_j > \alpha_j$, implying that a firm with a given market share will have higher passthrough when competing against a more responsive firm than otherwise.

3.2 Static One-Period Game With Adjustment Costs

I now return to the environment with positive adjustment costs and consider the model's ability to match the empirical findings that price duration is larger for more differentiated products and that prices change with significant synchronization. This model will be able to generate both of these comparative statics.

As seen in equation (3), there are two shocks that could lead a firm to change its price – a shock to its own production cost and a change in its competitor's price – and a host of conditions and parameters, such as the market share and the size of the adjustment cost, that influence this decision. To build intuition, I start by considering a one-period game where there is no price response from competitors and firms start in their flexible price equilibrium, with initial profits denoted by $\pi_j^+ = \pi_j^+(m^+, p_{-j}^+)$. From this point, if firm j foregoes price adjustment in the face of higher production costs, there is no change in revenue or demand, and the firm's profits will decline by exactly this cost change times the number of units:

$$d\pi_j^N = \pi_j\left(m_j^+ + dm_j, p_{-j}^+\right) - \pi_j^+ = -c_j dm_j = -c_j m_j \widehat{m_j},$$

where the superscript "N" stands for "non-adjustment." To consider the change in profits that would occur under adjustment (represented with "A") to this shock, I write the second-order approximation around the flexible price equilibrium just prior to a cost shock:

$$d\pi_j^A = \pi_j\left(m_j^+ + dm_j, p_{-j}^+\right) - \pi_j^+ \approx \frac{\partial \pi_j^+}{\partial m_j} dm_j + \frac{1}{2}\frac{\partial^2 \pi_j^+}{\partial m_j^2} dm_j^2.$$

The overall incentive to change prices, an object that implies shorter price durations as it gets

bigger, is approximated as the difference between the two: $d\pi_j^A - d\pi_j^N$.

I show in Appendix A that $\partial\pi_j^+/\partial m_j = -c_j$, and hence the first order terms for the change in profit with and without adjustment cancel. As a result, the approximated adjustment incentive is only the second-order term $\frac{1}{2}\frac{\partial^2\pi_j^A}{\partial m_j^2}dm_j^2 = \frac{1}{2}\Omega_j\widehat{m_j}^2$, where:

$$\Omega_j = (\varepsilon_j - 1)\, s_j \alpha_j cx, \tag{5}$$

and where cx denotes total distributor spending on the sector's inputs. After fixing manufacturer revenues, Ω_j can be written as the product of $(\varepsilon_j - 1)$ and α_j, as is the focus of Gopinath and Itskhoki (2010), which first derived such an expression in a similar model with monopolistic competition.

More differentiated goods in this model will not always have stickier prices because $d\Omega_j/d\rho$ cannot be unambiguously signed. To get a sense for the comparative static of duration with respect to degree of differentiation, I consider the following numerical exercise, plotted in Figure 1. I set initial productivity levels for the two firms to be equal, $m_j = m_{-j}$, and pick a symmetric and constant value for the adjustment cost $\phi = \phi_{j,t} = \phi_{-j,t}$. Under this configuration, the firms start with equal market shares. Starting from equilibrium in the flexible price model (denoted with the black plus sign), a firm observes its own cost shock and its competitor's price change and determines if adjustment merits payment of the fixed cost.

The left plot is drawn from the perspective of an input manufacturer in a highly differentiated sector, where shocks to its competitor's price and its own cost are represented with the horizontal and vertical axes, respectively. The right plot is the exact same, but for a less differentiated sector (with higher ρ). The red regions are then defined as the portions of the state space where a firm does not adjust prices and the boundaries can be thought of as s-S bands.

The scenario where a firm's production cost increases by 5 percent and the competitor raises prices by 10 percent is represented by a move upward from the black plus sign by 0.05 and to the right by 0.10. If such a move does not exit the red region, it means that given these shocks, a firm would not change its price. If such a move crosses the upper boundary into the "raise" region, it means the shocks are sufficiently large to warrant a price increase, even if facing an adjustment cost.

The first key observation is that the no-adjust region for both firms has negative slope. If a change in the second firm's price is large enough, it can induce the first firm to change prices, even if the first firm does not incur a shock to its marginal cost. This is the visual manifestation of the strategic complementarity in the model and is the force generating synchronization in the timing of price changes. Secondly, the vertical width of the band is wider for the more differentiated case. Given the degree of stickiness in the data, own-cost shocks are far more prevalent than competitor-price shocks and hence, the vertical width is the crucial determinant of stickiness. It is clear that any given cost shock is more likely to exit the red region, up or down, for the less differentiated good arm's length firm. Though one can find places in the parameter space where these results do not hold, they are far away from the most natural benchmarks such as symmetry and generally require significantly skewed productivity distributions in the sector.

4 Recursive Formulation and Solution

The previous sections' derivations rely on several simplifying assumptions or approximations, abstract from option value, and consider the occurrence of each shock and each firm's pricing decision only one at a time. In reality, firms have expectations about each other's responses to shocks and typically start periods away from their flexible price equilibrium. In this section, I move to a dynamic setting in order to address these shortcomings.

The monetary authority maintains a constant retail price level, $P_t = 1$, and thus fixes aggregate consumption $C_t = C$. This leaves four principal state variables in the system – the two manufacturing prices from the previous period and the two marginal costs in the current period. I bundle these four dimensions of the state space as $\Theta_t = \{p_{1,t-1}, p_{2,t-1}, m_{1,t}, m_{2,t}\}$. Most dynamics are generated by the fully observable shocks to the marginal cost of production for each firm. The other source of dynamics follows from the random adjustment cost $\phi_{j,t}$.

Firms follow pure strategies in price setting. For a given state $\{\Theta_t, \phi_{j,t}\}$, each firm j simultaneously chooses a unique price. As emphasized in Doraszelski and Satterthwaite (2010), due to the uncertainty about the competitor's adjustment cost, a firm generally does not know with certainty what strategy its competitor will play. Hence, from the perspective of firm $-j$, the probability that firm j changes prices in a given period is $\xi_j(\Theta) = \int \chi_j(\Theta, \phi_j) dG_j(\phi_j)$.

A Markov Perfect Equilibrium is defined as a set of pricing policies for each firm j, $p_{j,t} = p_j(\Theta_t, \phi_{j,t})$, where $p_{j,t}$ maximizes expected firm profits, consistent with consumer demand, and where each firm has correct expectations about the distribution of its competitor's prices across realizations of the competitor's adjustment cost.

Let $V_j(\Theta, \phi_j)$ denote the conditional values of the firm, after each has observed its own price adjustment cost. I define these value functions recursively as:

$$V_j(\Theta_t, \phi_{j,t}) = \max_{\widetilde{p}_j} \quad \xi_{-j}\pi_j(\widetilde{p}_j, \widetilde{p}_{-j}) + \left(1 - \xi_{-j}\right)\pi_j(\widetilde{p}_j, p_{-j,t-1}) \\ - \chi_j(\Theta_t, \phi_{j,t})\phi_j + \beta \int_{u_j} \int_{u_{-j}} V_j(\Theta_{t+1}) dF_j dF_j, \quad (6)$$

for each firm j. Here, it is easy to see the difficulty in modeling this type of strategic behavior – it requires solving a coupled system of Belman equations where each firm j's optimal policy depends on that of firm $-j$. The final term in (6) contains $V_j(\Theta') = \int V_j(\Theta', \phi_j) dG_j(\phi_j)$, the expected value function of firm j, conditional on being in state Θ' but before observing its adjustment cost (expectations here are taken only over uncertainty about the realization of this cost).

Following Doraszelski and Satterthwaite (2010), I integrate both sides of these Belman equations over all realizations of their respective adjustment costs and re-write the value function in equation (6), which is a function of five variables, as the expected value function, which is no longer a function of the adjustment cost:

$$V_j(\Theta_t) = \max_{\substack{\xi_j \in [0,1] \\ \widetilde{p}_j \in (0,\infty)}} \quad E\left[\pi_{j,t}\right] - \int_{\phi_j < G_j^{-1}\left(\xi_j(\Theta_t)\right)} \phi_j dG_j\left(\phi_j\right) + \beta \int_{u_j} \int_{u_{-j}} E\left[V_j(\Theta_{t+1})\right] dF_j dF_{-j}. \quad (7)$$

Expected profit, $E[\pi_j]$, is the probability weighted average across the four combinations of {adjust,no-adjust} $\times$ {adjust,no-adjust}, and the transition of the first two state variables is similarly defined in the expected continuation value. Formally:

$$E[\pi_{j,t}] = \sum_{\alpha_1=0}^{1} \sum_{\alpha_2=0}^{1} \xi_1^{\alpha_1} \left(1 - \xi_1\right)^{1-\alpha_1} \xi_2^{\alpha_2} \left(1 - \xi_2\right)^{1-\alpha_2} \pi_j \left(\left(\widetilde{p}_1\right)^{\alpha_1} \left(p_{1,t-1}\right)^{1-\alpha_1}, \left(\widetilde{p}_2\right)^{\alpha_2} \left(p_{2,t-1}\right)^{1-\alpha_2}, m_{j,t} \right),$$

and:

$$E\left[V_j(\Theta_{t+1})\right] = \sum_{\alpha_1=0}^{1}\sum_{\alpha_2=0}^{1} \xi_1^{\alpha_1}\left(1-\xi_1\right)^{1-\alpha_1}\xi_2^{\alpha_2}\left(1-\xi_2\right)^{1-\alpha_2} V_j\left(\left(\widetilde{p}_1\right)^{\alpha_1}\left(p_{1,t-1}\right)^{1-\alpha_1}, \left(\widetilde{p}_2\right)^{\alpha_2}\left(p_{2,t-1}\right)^{1-\alpha_2}, \cdot, \cdot\right).$$

Subject to the above system of demand, production, and cost shocks, the two firms play a non-cooperative dynamic game in pure Markov pricing strategies. I follow Midrigan (2010, 2011) and Miranda and Vedenov (2001) and use projection methods (collocation, specifically) to approximate the solution to this coupled system of Belman equations.[10] A detailed description of the solution algorithm is given in Appendix B. Figure 2 shows a sample plot (holding fixed the values for the competitor's previous price and current cost) of a policy function from the solution of the model. The vertical axis gives the conditional probability of a price change before observing the menu cost realization and the x- and y-axes give the firm's previous price and current cost. This plot makes clear that, despite the time-dependency added by the stochastic menu cost, the model preserves its state-dependent flavor. The probability of a price change fluctuates dramatically across the state space, even if it transitions more smoothly than the zero to one fluctuations in a standard state-dependent model.

5 Simulation Results

To assess the model's predictions for price duration, passthrough, and sychronization, I take the approximated policy functions and generate series of costs and prices for various ranges of the parameter space. The two-input structure of my model rules out treatment of the simulation as a true calibration exercise – I do not focus on quantitatively matching any moments, but rather, focus on reproducing the key qualitative features of the data on duration, passthrough, and synchronization.

I simulate the model for five sectors with varying elasticities of substitution, ρ. The period length is intended to represent one month and the discount factor is set at $\beta = 0.99$. I set a normal distribution for the monthly shock process, μ, with a standard deviation of 2.5 percent for both manufacturers, roughly that of the U.S. dollar to Euro exchange rate. Identical uniform distributions (with limited support) are used for each firm's adjustment costs such

[10] The techniques used are described in-depth in Miranda and Fackler (2002), which also provides an accompanying MATLAB toolbox (CompEcon) that was used extensively for this paper.

that the median duration magnitude roughly fits the level of stickiness in the international trade micro-data documented in Gopinath and Rigobon (2008).

In each sector, I plot results from a configuration generating equal average market shares, though the qualitative patterns I focus on do not change if I consider non-extreme deviations from symmetry such as shares of 60 and 40 percent or 75 and 25 percent.[11] I specify costs and aggregate demand parameters such that, across all sectors, the firms sell for the same average price and sell the same average quantity of goods. This ensures that differences in stickiness do not reflect differences in the size of the adjustment cost relative to the scale of business activity.

I set $\lambda = 0.75$. In the open-economy interpretation of the model, this parameter is consistent with a typical import share from OECD input-output tables and will, of course, scale down passthrough levels. This and other parameter values are summarized below:

η	β	γ	λ	σ	δ	ρ_{Min}	ρ_{Max}
2	0.99	0.5	0.75	0.025	0.985	4	12

5.1 Importance of Strategic Behavior

If modeling a sector with many similar manufacturers exposed to similar shocks, a model with atomistic firms and variable markups would be a preferable setup (and may in fact be the only computationally feasible one). If modeling an oligopolistic sector where firm-specific shocks matter for sectoral prices, however, a model of strategic firms playing a pricing game is preferable since the approximation made by atomistic firms using only sector-level variables is unlikely to be accurate. For example, imagine there are two key foreign suppliers of intermediate goods in a sector, one in Japan and one in Mexico. Depending on the market share configuration, the sector-level price index will be differentially sensitive to the scale of each firm-specific shock and the probabilities of each firm in fact changing its price.[12]

[11] "Average" values for various firm-level statistics are meaningful over long enough periods of time because cost shocks, though highly persistent, are mean reverting as $\delta < 1$.

[12] It is difficult to know how small firms must be for firm-specific shocks to no longer meaningfully influence sectoral prices. Here we consider a model with two strategic firms, where the importance of firm-specific shocks is greatest. There would be no conceptual difficulties in expanding this framework to 3-firm sectors (thereby having 6 states), though the simulation would clearly take much longer to code and run. Expanding to 4 firms and beyond would likely, at current technology levels, require a particularly fast computer or a particularly long time to run.

Figure 3 shows an example of the price and cost series generated by the simulation of this pricing game. The prices and costs are plotted against the left axis, while the probability of adjustment ξ_j is indicated by the shaded bars and is measured on the right axis. Consider the episode that occurs in the beginning of year 5. Firm j increases its price even though its own cost has been declining. This is labeled an "example of complementarity" because the price increase is clearly driven by the (correct) expectation that the other firm, it's competitor, would increase its own price. Firm j's decision to change prices reflects a calculation that considers the shocks, menu costs, market shares, and strategy of firm $-j$. By contrast, the equivalent price setting decision in most models with a continuum of atomistic firms would only reflect information on j's own marginal cost and an approximation of other firms' behavior based on sectoral variables. The resulting decision would likely be very different from the optimal strategy calculated here.

5.2 Duration: Empirics and Simulation

Results in the empirical literature suggest that prices of more differentiated goods change less frequently. For example, Gopinath and Rigobon (2008) show in their Table IV that the mean frequency of price change for reference priced (i.e. undifferentiated) goods is more than twice that of differentiated goods. "Raw goods", a highly substitutable category, is the least sticky in Bils and Klenow (2004) while "medical care," presumably highly differentiated, is the most sticky. Nakamura and Steinsson (2008) show that less differentiated goods like "unprocessed food" or "vehicle fuel" change prices far more often than more differentiated products like "processed food" or "services". The solid line in Figure 4 plots the median spell-weighted duration for firms with equal average prices and quantities but with varying elasticities of substitution (ρ). Consistent with equation (5), for the majority of reasonable parameter values and market share configurations, price duration or stickiness decreases as goods become less differentiated.

Equation (5) also suggests that, conditional on the elasticity of demand ε_j, price duration will decrease with the rate of passthrough α_j. As can be seen in equation (4), the firm faces a constant elasticity of demand in the limit as η approaches ρ, implying full passthrough and therefore lower stickiness, other things equal. The dashed line in Figure 4 plots durations from simulations of otherwise equivalent CES firms, where $\rho = \eta$ is set to match the average

elasticity of demand faced by the variable markup firms. As suggested by the static analysis in Section 3.2, conditional on the same average demand elasticity, firms with variable markups exhibit greater stickiness.

5.3 Passthrough: Empirics and Simulation

Many recent papers have demonstrated that, even conditional on price adjustment, cost passthrough is less than 1, including Gopinath, Itskhoki, and Rigobon (2010), Burstein and Jaimovich (2009), and Fitzgerald and Haller (2009). To capture this concept in the simulated data, I consider the φ coefficient from the pooled regression:

$$\Delta \ln p_{j,t_j-t_j^{-1}} = \alpha + \varphi \Delta \ln e_{j,t_j-t_j^{-1}} + \varepsilon_{j,t_j}, \tag{8}$$

where t_j and t_j^{-1} are good specific and respectively denote the times of the most recent and penultimate price changes. Only non-zero price changes are included in the regression, and $\Delta \ln p_{j,t_j-t_j^{-1}} = \ln(p_{j,t_j}/p_{j,t_j^{-1}})$ denotes the size of the most recent price change and $\Delta \ln e_{j,t_j-t_j^{-1}} = \ln(e_{j,t_jj}/e_{j,t_j^{-1}})$ denotes the accumulated change in the cost shock from the time of previous price change to the time of the most recent change. The solid line in Figure 5 plots this passthrough coefficient (which, given it is run on simulated data, is very precisely estimated) for the variable elasticity of demand firms in our model. As in the empirical results throughout the literature, passthrough rates, even after price adjustment, is clearly incomplete.[13]

As noted in Table 1, variable markups are required to generate incomplete marginal cost passthrough. To see this, the dashed line in Figure 5 also shows estimates of φ from simulations run on otherwise equivalent firms with $\rho = \eta$ and that therefore face a constant elasticity of demand. Passthrough rates, unsuprisingly, roughly equal λ across all sectors. Since the elasticity of marginal cost to exchange rate shocks equals λ, this verifies that unlike the baseline firms, these CES firms exhibit complete passthrough of marginal cost shocks.[14]

[13] Exchange rate passthrough estimates in the micro-data literature are quite small and range from about 10 percent to about 50 percent. As with most of the passthrough literature, this model's average rate of passthrough of 52 percent is thus too high.

[14] The CES firms do not exhibit truly complete passthrough exactly equaling λ since their dynamic pricing decisions reflect the asymmetric payoffs from the possibility of having too high or two low a price in future months.

5.4 Synchronization: Empirics and Simulation

Finally, Cavallo (2011) and Midrigan (2011) demonstrate that price changes are synchronized. There is no standard measure used to quantify price change synchronization. Here, I observe the percentage of simulated months in which both manufacturers' prices change and compare it to the percentage that would be randomly generated. For instance, if firm 1 changes its prices every d_1 months, and firm 2 does so every d_2 months, zero synchronization would imply the existence of months with two prices changes about $100/(d_1 d_2)$ percent of the time. Hence, I measure synchronization in the simulated data as a ratio ("synchronization ratio") of the frequency of months with two price changes to the frequency that would be expected with randomly timed changes. The vast majority of time-dependent models would, for example, generate ratio values of one. Values greater than one suggest synchronization in the data.

The solid line in Figure 6 shows the synchronization ratio across sectors with varying elasticities of substitution. The ratios are all greater than one, suggesting that the model produces price change synchronization. Further, consistent with the discussion of Figure 1, less differentiated sectors exhibit greater degrees of synchronization in price setting.

Finally, I again take the average elasticity of demand faced by firms in these sectors and compare with synchronization in sectors with two CES firms that face a constant elasticity of demand at that save average level. The dashed line plots the synchronization ratios for these firms, which is always below the value for the baseline model and is roughly constant at a value of one. The baseline model's success at generating synchronization requires strategic complementarity arising from a variable elasticity of demand.

6 Application: Related Party Trade

I have stressed that modeling price setting as a game between a small number of firms is more important in settings in which there are large firms facing different shocks and with heterogenous market shares. Another important form of heterogeneity that can influence sectoral price behavior stems from differences in the vertical structure of trade. One benefit of my model is that it can be easily extended to consider the behavior of trade prices used by related parties when conducting intrafirm trade. Approximately 40 percent of all imports into the United States are classified as related party transactions by the Bureau of Labor

Statistics (BLS). Empirical results in Bernard et al. (2006), Hellerstein and Villas-Boas (2010), and Neiman (2010) show that intrafirm prices exhibit less stickiness, greater exchange rate passthrough, and lower synchronization.[15]

I extend the model to consider the case in which one product is assembled from a related party, which sells its input to a wholly owned subsidiary (or parent). I do not consider the case in which both firms are related parties as this would render the setup, in which manufacturers do not coordinate price-setting with each other, unrealistic. Distributors that purchase from a related party also purchase from arm's length suppliers, a feature with empirical support.[16] The upstream related party supplier attempts to avoid double marginalization and sets trade prices to approximately follow marginal cost. Accordingly, intrafirm price setting is primarily inward looking and responds less to competitors' prices, which leads to less price synchronization and greater passthrough of marginal cost shocks. Further, a second order approximation to the intrafirm price adjustment incentive reveals that the duration of a firm's price is related positively to its market share and negatively to its cost of goods sold and to its rate of passthrough. On average, related party duration is therefore lower because intrafirm passthrough and cost of goods sold, conditional on market share, will both be higher.

6.1 Price Setting: Intrafirm Trades

Vertically integrated firms aim to maximize overall profits – the sum of its profits at the manufacturer and distributor levels – as follows. The manufacturing firm (or a separate headquarters division) instructs the distributor to take input prices as given, and to purchase from the arm's length or related party manufacturer in whatever way maximizes distributor profits. This should not be interpreted as if the distributor is naive of the ownership structure or acts myopically, but rather, is simply following the pricing mechanism designed by the integrated firm. As part of this mechanism, the manufacturer knows how the distributors will act and thus chooses prices in order to maximize the expected present value of all future integrated profits, after subtracting price adjustment costs. Anecdotal evidence suggests that

[15] The share of intermediate good transactions in intrafirm trade differs from that in arm's length trade. The results in Neiman (2010), however, also hold when restricting the analysis to intermediate goods (based on 1-digit end-use codes).

[16] Bernard et al. (2007) shows that the vast majority of firms that import from related parties also do so from arm's length suppliers.

the essence of this pricing mechanism is used by actual companies.[17]

In the absence of any frictions between the upstream and downstream units, this would not be the profit maximizing pricing mechanism. For instance, if the distributor is aware of a marginal cost shock to the upstream related party, then it would optimally change its own retail price even if the transfer price remained unchanged. However, I rule out this possibility by assuming that the distributor does not itself observe the manufacturer's marginal cost shock and that the manufacturer would incur the adjustment cost if it communicated this information to the distributor. This is consistent with the above-described interpretation of an adjustment cost.

Without loss of generality, I assume the related party manufacturer in this case supplies the second manufactured input. The integrated firm's operating profits are:

$$\pi_2 = \left[\pi_2^{Distributor}\right] + \left[\pi_2^{Manufacturer}\right] = pc - p_1 c_1 - m_2 c_2, \tag{9}$$

and can be re-written as:

$$\pi_2 = CP^{\eta}\left(\frac{\eta}{\eta-1}\right)^{1-\eta}\left[\gamma^{\rho}{p_1}^{1-\rho} + (1-\gamma)^{\rho}{p_2}^{1-\rho}\right]^{\frac{1-\eta}{1-\rho}} - CP^{\eta}\left(\frac{\eta}{\eta-1}\right)^{-\eta}\left[\gamma^{\rho}{p_1}^{1-\rho} + (1-\gamma)^{\rho}{p_2}^{1-\rho}\right]^{\frac{\rho-\eta}{1-\rho}}\left[\gamma^{\rho}p_1^{1-\rho} + (1-\gamma)^{\rho}p_2^{-\rho}m_2\right].$$

Vertically integrated firms maximize the present value of real profits, less real adjustment costs $\phi_{j,t}/P_t$, and maximize an expression equivalent to (2).

6.2 Flexible Prices: Intrafirm Case

I now revisit the flexible price setting from Section 3.1 and consider related party trade. In this setting with zero adjustment costs, pricing for related parties is simple. Comparing the expression for distributor per-period profits in equation (1) with that for the integrated

[17]The managing director of a consulting firm specializing in transfer pricing told me of the case of a large multinational company that evaluates upstream manufacturing managers on their ability to minimize production costs, without any link to the upstream unit's profits (the company delegates the determination of transfer prices, but not retail prices, to a separate group that aims to maximize overall firm profitability). The consultant described another integrated relationship in which the downstream unit, by design, made purchases without even knowing which suppliers were related parties and which were arm's length firms. Both anecdotes support the idea in the model that transfer prices may be both allocative and designed to maximize the sum of upstream and downstream profits.

firm in equation (9), it is clear that in order for the solution to the distributor's problem to always equal the solution to the integrated firm's problem, related party manufacturers should charge their marginal cost: $p_j = m_j$ if $j = RP$ (where I now use $j = RP$ to denote the related party manufacturer and $j = AL$ to denote the arm's length manufacturer). As discussed in Hershleifer (1956), the transfer occurs at marginal cost because the firm wants to use inputs as efficiently as possible in generating the final good, since the final good consumer is the only real customer. Above, differentiated good arm's length firms were shown to charge higher markups than homogenous good arm's length firms. Combined with marginal cost transfer pricing, this implies that intrafirm prices of equivalent goods will be lower than arm's length prices, and the difference should be larger for more differentiated goods. This is precisely the result found empirically in Bernard et al. (2006).

With no adjustment costs, related parties will fully pass through the portion of the shock μ_j that changes its unit cost. In particular:

$$\widehat{p_j} = \widehat{m_j} = \lambda de_j \approx \lambda \mu_j \text{ if } j = RP, \tag{10}$$

where the approximation becomes an equality as $\delta \to 1$. Hence, intrafirm passthrough equals λ, which corresponds to complete passthrough of marginal cost shocks.

In this sense, the related party manufacturer is less concerned with the arm's length firm and is focused entirely inward, on its own marginal cost. In the dynamic model with adjustment costs, related parties will not strictly price at marginal cost because the firm must weigh whether it prefers to be slightly above or below its ideal flexible price in future periods where a price change is not warranted. It will remain true in the model, however, that related party passthrough is very close to λ. This implies, consistent with the empirical results in the literature, that intrafirm passthrough will be higher than arm's length passthrough. Further, the competitor firm's price is absent from the pricing equation (10), so this model will generate less price synchronization in sectors with related party trade, also consistent with the data.

6.3 Static One-Period Game with Adjustment Costs: Intrafirm Case

Finally, I now revisit the static one-period exercise from Section 3.2 where firms begin in their flexible price equilibrium but must now pay an adjustment cost to change prices. I show that

the model can match the empirical results that intrafirm prices change more frequently and with less synchronization.

Appendix A shows that the related party pricing structure leads to an expressions for the approximate adjustment incentive $\Omega_{RP,j}$:

$$\Omega_{RP,j} = \varepsilon_j s_j cx. \tag{11}$$

The difference between the related party expression in (11) and the arm's length expression in (5) reflects the fact that a firm's cost of goods sold, $COGS_j = c_j m_j$, scales each firm's incentive to change prices for a given percentage cost shock. Since arm's length firms charge a markup and related parties do not, the cost of goods sold is related differently to market shares and elasticities for the two firms.

Substituting $COGS_{RP,j} = s_{RP,j} cx$ and $COGS_{AL,j} = s_{AL,j} cx \frac{\varepsilon_j - 1}{\varepsilon_j}$ into expressions (11) and (5), I can write the incentives as $\Omega_{AL,j} = \varepsilon_j \alpha_j COGS_{AL,j}$ and $\Omega_{RP,j} = \varepsilon_j COGS_{RP,j}$. This gives the intuition for why related party duration will be shorter, conditional on the market share, and all other things equal. The market share uniquely determines the demand elasticity ε_j, and given the related party charges no markup, its cost of goods sold must be higher. The variable markup component of passthrough, α_j, is strictly less than one, so $\Omega_{RP,j} > \Omega_{AL,j}$.

In Appendix A, I demonstrate for the two-firm case that $s_{AL,j} < \eta / (2\eta - 1) = \overline{s_{AL}}$ is a sufficient, though not necessary, condition for $\Omega_{RP} > \Omega_{AL}$. Note that as $\eta \to 1$, $\overline{s_{AL}} \to 1$, and there is no portion of the parameter space where the approximation suggests stickier related parties, regardless of initial productivities. In the model's other extreme, as $\eta \to \infty$, $\overline{s_{AL}} \to 1/2$. Given arm's length markups exceed those of related parties, this implies that with equal productivities, related parties are less sticky everywhere in the parameter space. Numerical exercises suggest that for any given η, an increase in ρ increases the maximum arm's length market share below which its prices will be sticker. For plausible parameter values in this model, the threshold is at least two-thirds, and often much higher. This absolute level will of course decrease in a multifirm model, but the requirement that arm's length firms hold a significantly larger market share in order to be less sticky will generally hold, regardless of the number of firms. Hence, this static model generally predicts less sticky intrafirm prices.

Figure 7 shows s-S bands similar to those shown for the arm's length case in Figure 1,

but instead of comparing across elasticities of substitution, it compares the pricing decision of an arm's length firm (left) to that of a related party (right) for a given sectoral elasticity. Again, I set initial productivity levels equal, $m_{AL} = m_{RP}$, and pick a uniform value for the adjustment cost ϕ. This implies market shares will differ, but plots from the case of equal market shares are qualitatively the same.

First, note that the no-adjust region for the related party is essentially flat. This means that, when integrated firm prices are close enough to their flexible price target, there is no price change from the competitor (arm's length) firm that could induce the related party firm to change its price. Only as one moves vertically away from the horizontal line $p_{RP} = 1$ does the region begins to have any curvature. This follows because the result that related party price setting is inwardly focused is only strictly true when at the flexible price equilibrium. In this sense, Figure 2 helps one visualize why the model is able to produce greater synchronization among arm's length trades than intrafirm trades. Secondly, the vertical width of the band is smaller for the related party case, indicating less price stickiness and corroborating the results from the second order approximation.

6.4 Simulation Results

I now discuss results of a simulation of the sector with related party trade, maintaining the same parameter values as in the arm's length case discussed above. The simulations will produce shorter related-party price spells, higher related party passthrough, and lower synchronization in sectors with related parties. A natural alternative assumption might be that intrafirm adjustment costs are lower than arm's length adjustment costs. Lower related party adjustment costs would certainly generate lower stickiness, but on their own cannot explain the results on passthrough and synchronization.

I consider three cases: In the first, I set the firms' average market shares equal ($s_j = s_{-j}$); in the second, I set productivities equal ($m_j = m_{-j}$); and in the third, I set the firms' average cost of goods sold to be equal ($c_j m_j = c_{-j} m_{-j}$). These scenarios imply the related party's market share will be equal, larger, and smaller, respectively, than that of the arm's length firm. As above, I vary the sectoral elasticity of substitution across sectors.[18]

[18] Unlike the baseline simulations, the least differentiated sector I consider for the related party case has $\rho = 10$. The numerical routine often failed to converge for values higher than this.

Neiman (2010) shows that related party prices are stickier than arm's length prices in the same sector. Panel A of Figure 8 gives the averages of median spell-weighted duration across sectors for the arm's length and related party firms from their simulation prices. Intrafirm duration is graphed on the left bars, which are significantly higher for all three market share configurations, consistent with the data and the analysis in Section 6.3.

Bernard et al. (2006), Hellerstein and Villas-Boas (2010), and Neiman (2010) have shown that measures of exchange rate passthrough are higher for intrafirm than for arm's length price changes. To allow for comparisons of arm's length firms and related parties, I re-simulate the sector with two arm's length firms choosing productivities such that one of the arm's length firms in the baseline sector has both equal demand and market share as the related party firm in the hybrid sector. Panel B of Figure 8 plots the average passthrough coefficient across sectors from regressions of (8) for these otherwise equal arm's length and integrated firms. As in the data, arm's length conditional passthrough is consistently and significantly below that of related parties. Across sectors and market share configurations, passthrough to intrafirm prices is approximately λ, corresponding to complete passthrough of marginal cost shocks.

Finally, Panel C of Figure 8 compares the average synchronization ratios for all sectors with intrafirm trade and the re-simulated sectors with two arm's length firms. The analytics and static exercise in Section 6 indicated that, all things equal, there will be less synchronization in hybrid sectors with a related party. The hybrid sector exhibits less average synchronization for all three market share configurations, consistent with the evidence in Neiman (2010) that related party price changes are less synchronized.

6.5 Concerns about Transfer Pricing Data

Above, I compared the model-generated dynamics of intrafirm prices with the empirical patterns documented in Bernard et al. (2006), Hellerstein and Villas-Boas (2010), and Neiman (2010). It is not clear, however, that the data used in those analyses reflect allocative prices as opposed to accounting constructs used to achieve other goals, such as shifting income to lower-tax regimes. Neiman (2010) argues this is not a large problem in the BLS data by showing that the differential duration, passthrough, and synchronization patterns hold both in the sub-sample where the exporter's tax rates are highly similar to those in the U.S. and in the sub-sample where the tax rates are highly different. Bernard et al. (2006), however, interpret

their results from Census Bureau and Customs Bureau data as following in large part from tax-motivated transfer pricing. Given the BLS data is, unlike the Census and Customs data, explicitly separated from the taxing authority, both views may be correct.

If readers nonetheless believe that the intrafirm prices included in both of these data sets are not allocative, the ability of the model to match empirical patterns is not interesting. Even for these readers, however, the patterns generated from the simulated model should serve as a theoretical benchmark for how dynamics in sectors with significant vertical integration will differ from sectors without related party transactions. For example, there is no data I am aware of that includes transfer prices between related parties in the same country. In the absence of an empirical characterization of the dynamics of these prices, it is useful to note that the simulations above are suggestive that domestic industries with more vertical integration should exhibit less real and nominal rigidity.

7 Conclusion

A large number of recent empirical studies have documented new facts on stickiness, cost passthrough, and synchronization in final good and traded intermediate prices. Arm's length price stickiness is heterogenous and decreases with the elasticity of demand for a good. Incomplete cost passthrough is not simply a function of nominal rigidities and persists even after prices are changed. There is evidence of bunching in the timing of price changes. Further, studies that consider transactions between related parties have found that intrafirm stickiness and synchronization are lower and passthrough is higher. These facts present challenges to traditional pricing models in the open- and closed-economy macroeconomics literature. I write a state-dependent model where strategic firms set intermediate good prices that can be used to describe both arm's length and intrafirm pricing strategies and is capable of delivering all these empirical patterns.

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Tables

	(i)	(ii)	(iii)	(iv)
Model Features:				
Time/State Dependent:	Time	State	State	State
Elasticity of Demand:	Constant	Constant	Variable	Variable
Atomistic/Strategic Behaviror:	Atomistic	Atomistic	Atomistic	Strategic
Can it Be Used To Generate:				
Endogenous Stickiness that Varies with the Elasticity of Demand?	**N**	**Y**	**Y**	**Y**
Incomplete Marginal Cost Passthrough after a Price Change?*	**N**	**N**	**Y**	**Y**
Synchronized Price Changes Across Firms				
In reponse to aggregate shocks	**N**	**Y**	**Y**	**Y**
In response to firm-specific shocks	**N**	**N**	**N**	**Y**

*Assuming the marginal cost shock is permanent

Table 1: Ability of Classes of Models to Match Empirical Features

Figures

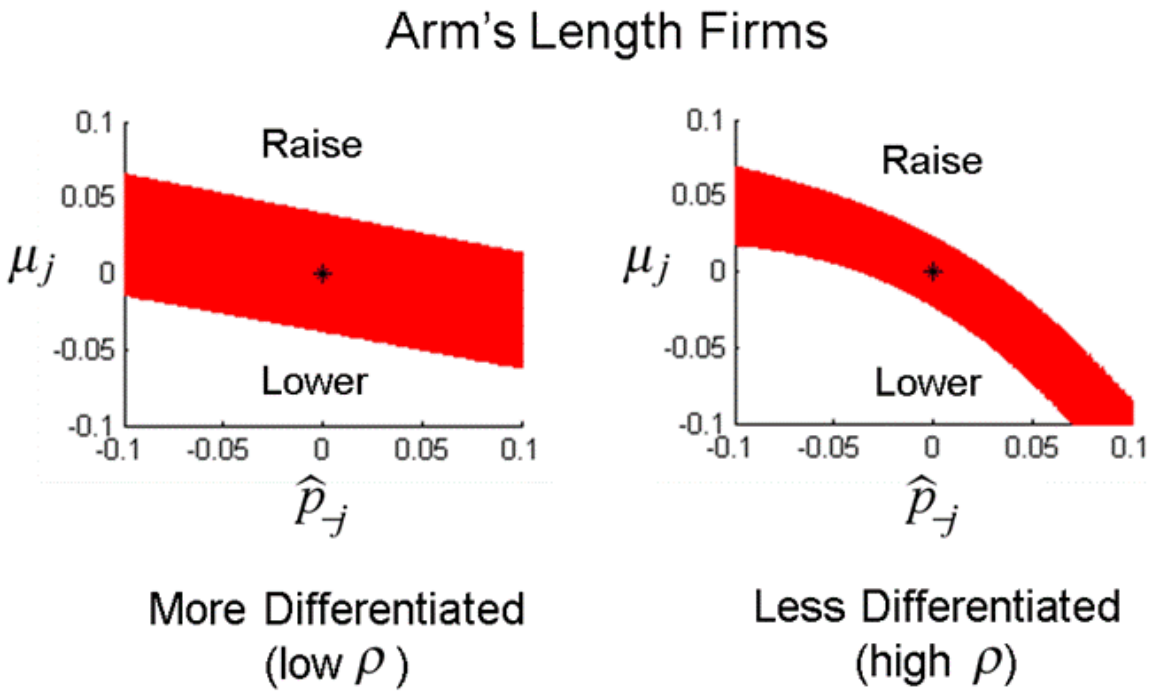


Figure 1: Arm's Length No-Adjust Regions by Elasticity of Substitution

Notes: Red regions define s-S bands within which a firm will not change prices. Movement along the vertical axes represents a percentage shock to a firm's own production cost, and movement along the horizontal represents a percentage shock to the competitor's price. No-adjust regions are calculated assuming there is no response from the competitor. The change from a thick band for differentiated sectors to a thin one for less differentiated sectors will generate heterogeneity in stickiness. Shocks will push the black marker outside of the red band more frequently in the right plot, leading to less stickiness. Both bands have a slope, indicating strategic complementarities and generating synchronization in price changes.

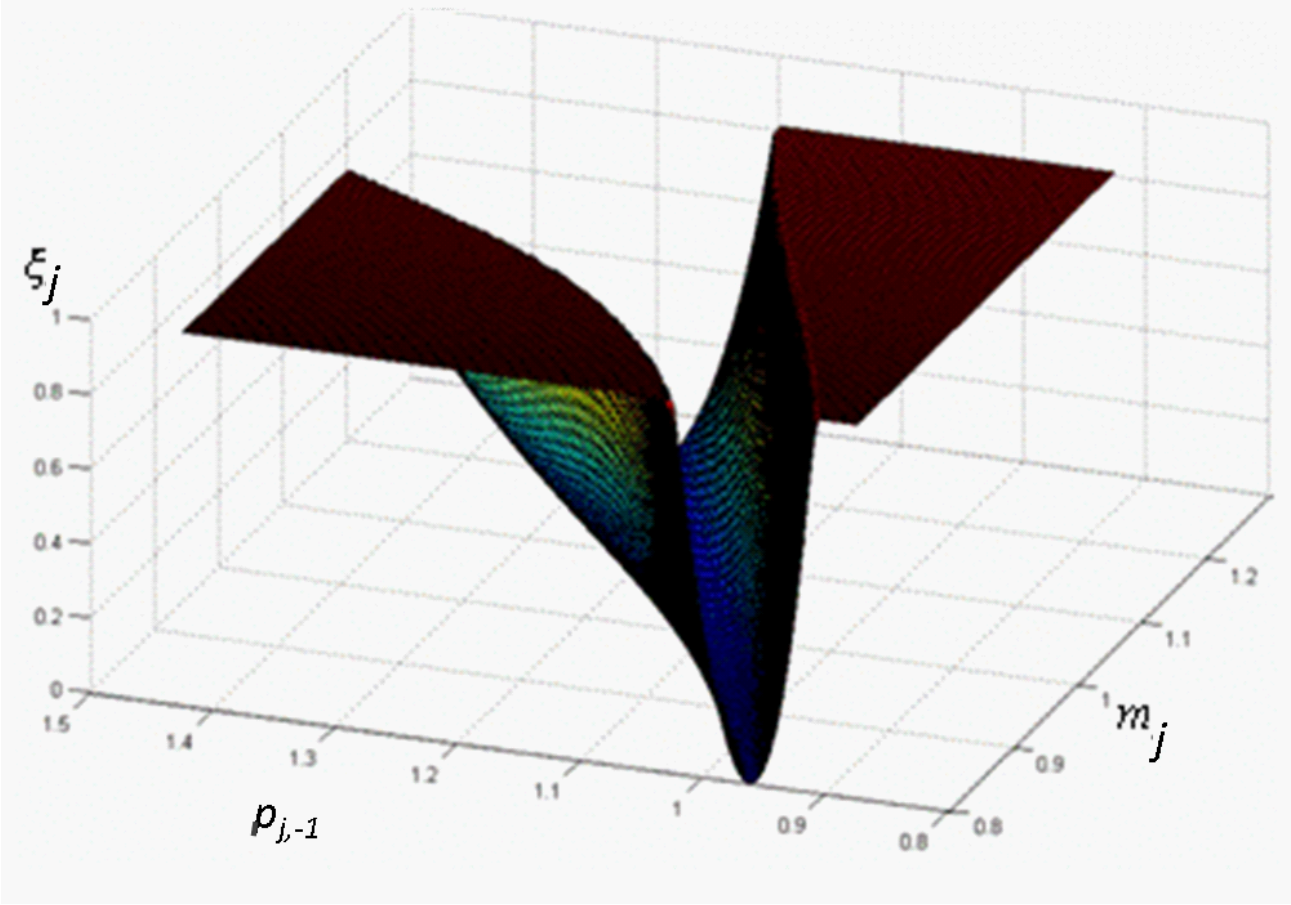


Figure 2: Sample Simulated Policy Function

Notes: Fixing particular values for the competitor's previous price and current cost, this is a sample policy functions where the vertical axis gives the conditional probability of a price change before observing the menu cost realization and the x- and y-axes give the firm's previous price and current cost. Though the random adjustment cost adds some time-dependence to the problem, this plots shows that the solution retains a state-dependent flavor as the probability of adjustment changes sharply with the state. The constant probability of adjustment in a Calvo model, for instance, would appear above as a flat plane.

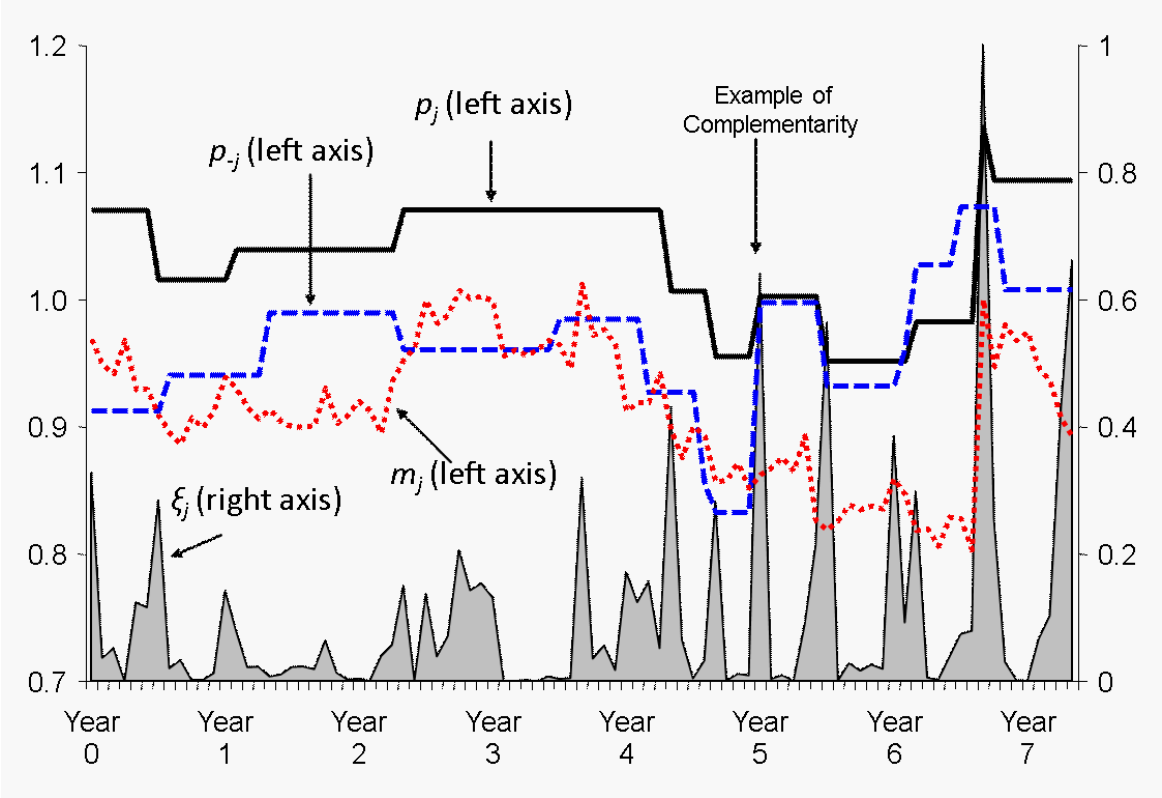


Figure 3: Strategic Complementarity in Model-Generated Data

Notes: This is a sample of the simulated price and cost data (lines, left axis) and probability of adjustment (shading, right axis) for a given sector. p_j and p_{-j} are the manufacturers' prices, m_j is the cost of arm's length firm j, and ξ_j is the probability of adjustment immediately prior to observing the month's adjustment cost. In the start of year 5, firm j increases its price even though its own cost has clearly been declining. This is labeled an "Example of Complementarity" because the price increase is clearly driven by the (correct) expectation that the competitor would increase its own price.

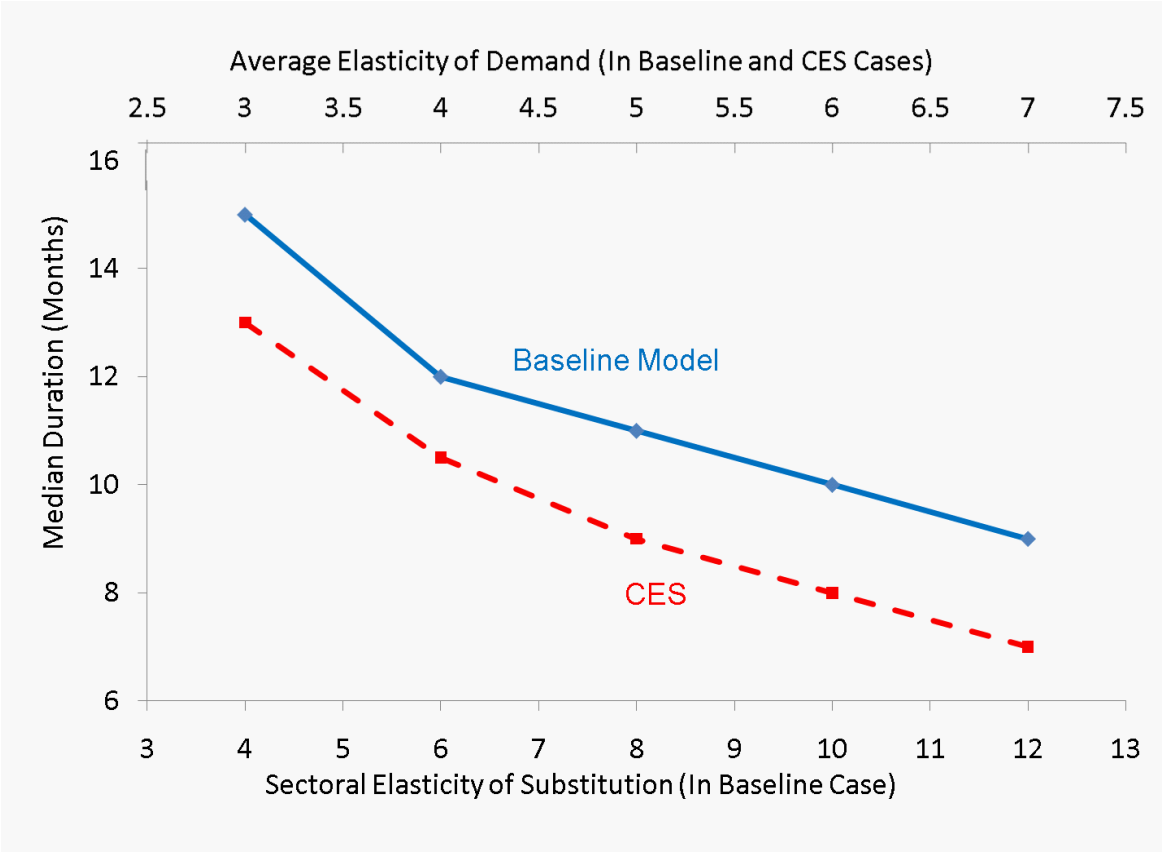


Figure 4: Simulated Spell-Weighted Median Duration

Notes: Results from simulation detailed in Section 5 and Appendix B. Both the state-dependent baseline model and the state-dependent CES model generate stickiness that decreases with the elasticity of demand. Consistent with the analysis in Section 3.2, other things equal, the CES case exhibits less stickiness. The elasticity of demand in the baseline case is the average of the sectoral elasticity of substitution ρ and the cross-sector elasticity $\eta = 2$. The CES case sets $\rho = \eta$. The horizontal axis atop the figure gives the average elasticity of demand for both models.

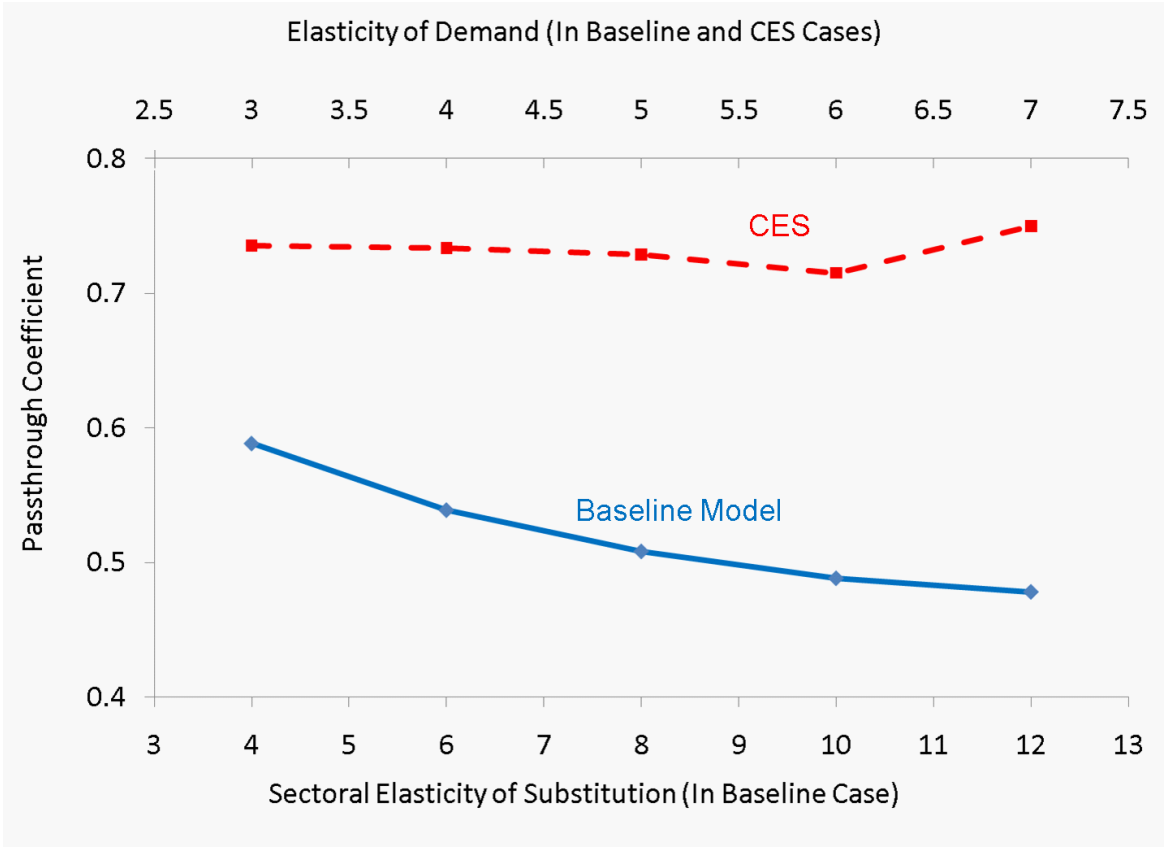


Figure 5: Simulated Passthrough

Notes: Results from simulation detailed in Section 5 and Appendix B. The state-dependent baseline model generates incomplete passthrough, even conditional on a price change. The state-dependent CES model approximately exhibits full passthrough. The elasticity of demand in the baseline case is the average of the sectoral elasticity of substitution ρ and the cross-sector elasticity $\eta = 2$. The CES case sets $\rho = \eta$. The horizontal axis atop the figure gives the average elasticity of demand for both models.

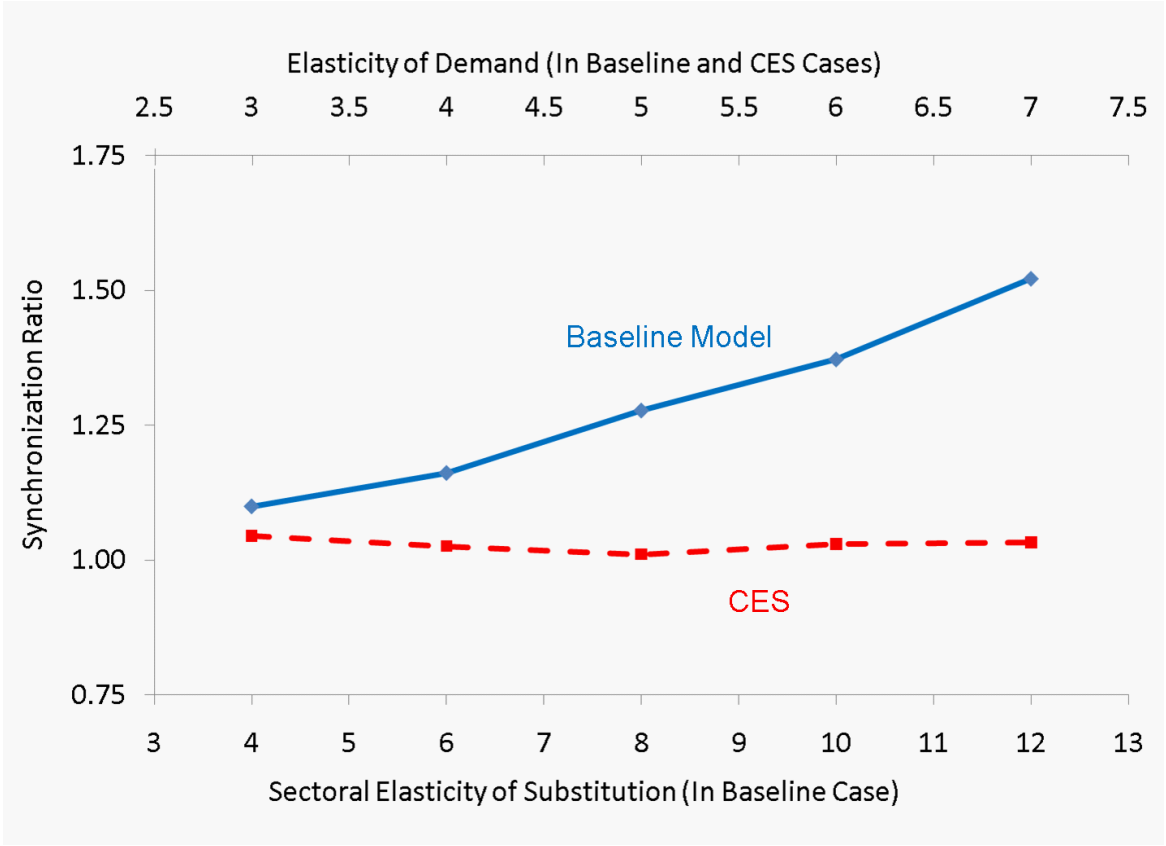


Figure 6: Simulated Synchronization

Notes: Results from simulation detailed in Section 5 and Appendix B. The state-dependent baseline model generates synchronization (represented by a synchronization ratio greater than 1). As expected, this synchronization is more pronounced in less differentiated sectors. Sectors with CES firms all have synchronization ratios very close to one indicating that, also as expected, they have no synchronization. The elasticity of demand in the baseline case is the average of the sectoral elasticity of substitution ρ and the cross-sector elasticity $\eta = 2$. The CES case sets $\rho = \eta$. The horizontal axis atop the figure gives the average elasticity of demand for both models.

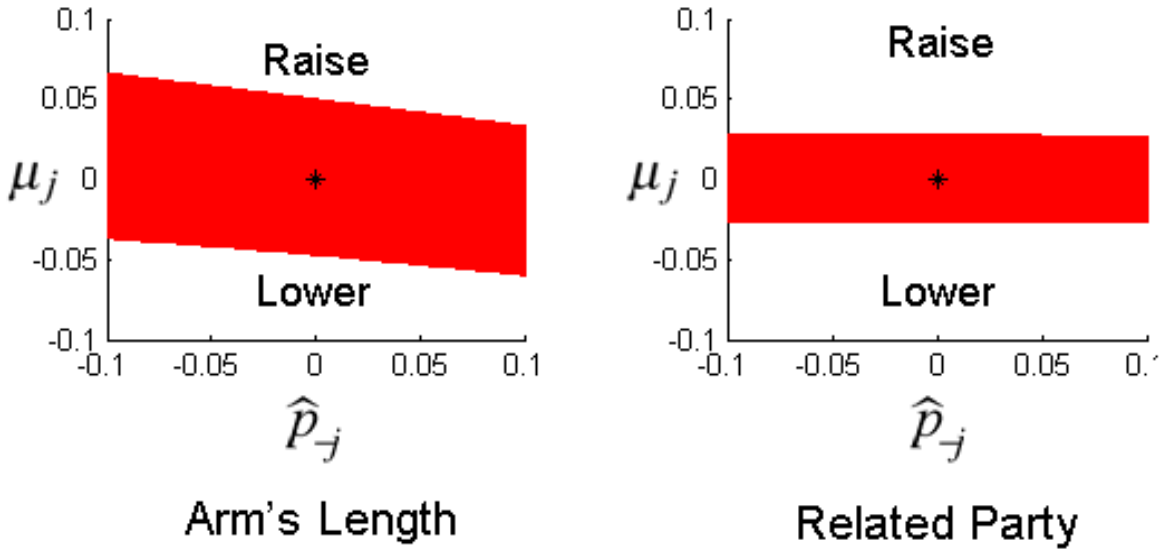
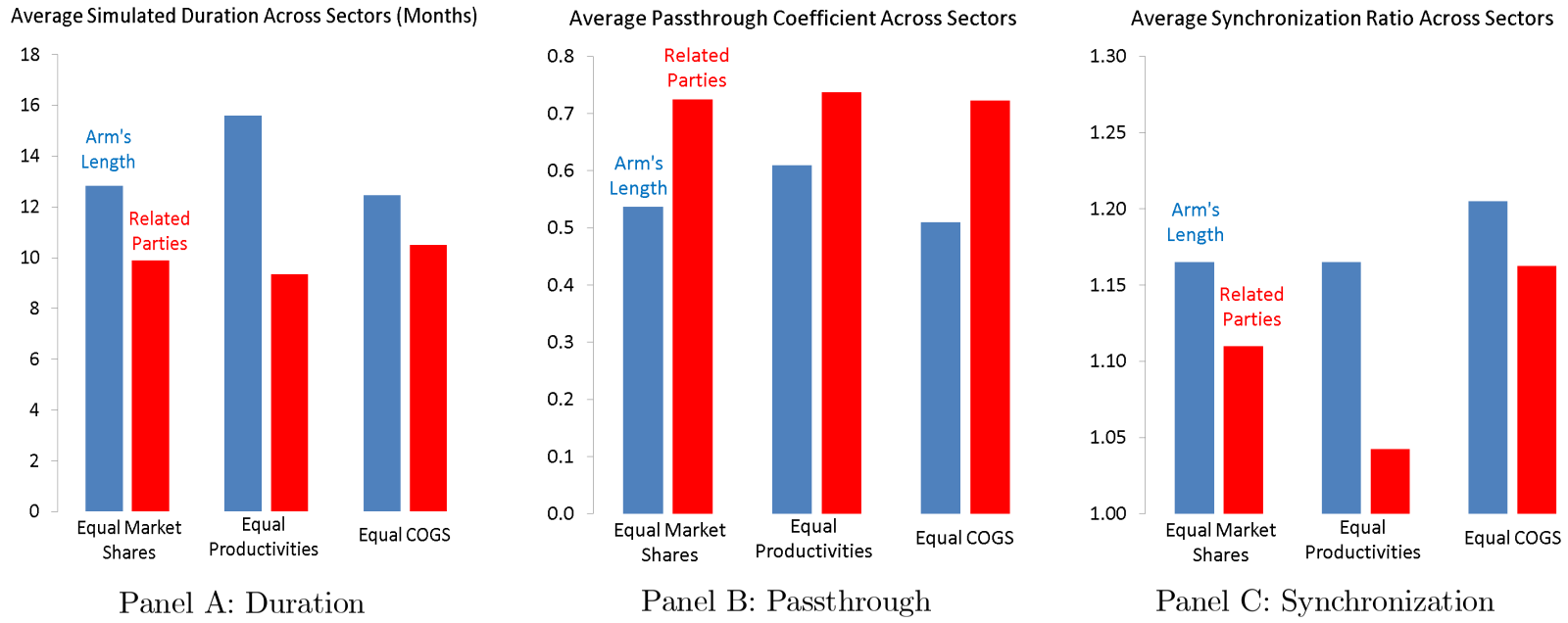


Figure 7: Arm's Length and Related Party No-Adjust Regions

Notes: Red regions define s-S bands within which a firm will not change prices. Movement along the vertical axes represents a percentage shock to a firm's own production cost, and movement along the horizontal represents a percentage shock to the competitor's price. No-adjust regions are calculated assuming there is no response from the competitor. The change from a thick band for the arm's length firm to a thinner one for related parties will generate lower related party stickiness. Further, only the arm's length band has a slope, indicating that related parties will be less influenced by strategic complementarities and may dampen price change synchronization.



Panel A: Duration

Panel B: Passthrough

Panel C: Synchronization

Figure 8: Simulated Price Dynamics for Arm's Length and Intrafirm Prices

Notes: Results from simulation detailed in Section 6 and Appendix B. To allow for comparisons of passthrough and synchronization for arm's length and related party firms, I re-estimate the structure with two arm's length firms, matching demand and market shares for the first firms in the two settings. I take simple averages of each statistic across sectors with ρ ranging from 4 to 10 and report this average for each of the three market share configurations. The bar chart shows that, consistent with the analysis of Section 6, related party prices change more frequently, exhibit higher passthrough, and result in lower synchronization in their sector.

Appendix A: Additional Calculations and Proofs

This appendix gives details for several of the calculations made in the text.

Claim 1 *We wish to show:*

$$\Omega_{AL} = \frac{s_{AL}\varepsilon\left(\varepsilon-1\right)^2}{\varepsilon\left(\varepsilon-1\right)+\left(\rho-\eta\right)\left(\rho-1\right)s_{AL}\left(1-s_{AL}\right)}cx.$$

With flexible prices, the arm's length firm's profits can be written as:

$$\pi_{AL}^{A} = \frac{1}{\varepsilon}c_{AL}p_{AL}.$$

Partially differentiating with respect to the optimal arm's length flexible price gives:

$$\begin{aligned}
\frac{\partial\pi_{AL}^{A}}{\partial p_{AL}} &= -\frac{1}{\varepsilon^2}\frac{\partial\varepsilon}{\partial p_{AL}}c_{AL}p_{AL}+\frac{1}{\varepsilon}\frac{\partial c_{AL}}{\partial p_{AL}}p_{AL}+\frac{1}{\varepsilon}c_{AL} \\
&= -\frac{1}{\varepsilon^2}\frac{\partial\varepsilon}{\partial s_{AL}}\frac{\partial s}{\partial p_{AL}}c_{AL}p_{AL}+c_{AL}\left(\frac{1-\varepsilon}{\varepsilon}\right) \\
&= -\frac{1}{\varepsilon^2}c_{AL}\left[\varepsilon\left(\varepsilon-1\right)+\left(\rho-\eta\right)\left(\rho-1\right)s_{AL}\left(1-s_{AL}\right)\right] \\
&= -c_{AL}\left(\frac{\partial p_{AL}}{\partial m_{AL}}\right)^{-1}.
\end{aligned}$$

This implies that we can write: $\frac{\partial\pi_{AL}^{A}}{\partial m_{AL}} = \frac{\partial\pi_{AL}^{A}}{\partial p_{AL}}\frac{\partial p_{AL}^{A}}{\partial m_{AL}} = -c_{AL}$. *Differentiating again, we get:*

$$\begin{aligned}
\frac{\partial^2\pi_{AL}^{A}}{\partial m_{AL}^2} &= -\frac{\partial c_{AL}}{\partial p_{AL}}\frac{\partial p_{AL}}{\partial m_{AL}} \\
&= \frac{c_{AL}}{p_{AL}}\frac{\varepsilon^3}{\varepsilon\left(\varepsilon-1\right)+\left(\eta-\rho\right)\left(1-\rho\right)s_{AL}\left(1-s_{AL}\right)}.
\end{aligned}$$

Substituting into the form: $\frac{1}{2}\frac{\partial^2\pi_{AL}^{A}}{\partial m_{AL}^2}\left(dm_{AL}\right)^2 = \frac{1}{2}\Omega_{AL}\widehat{m}^2$ *demonstrates the claim.*

Claim 2 *We wish to show:*

$$\Omega_{RP} = s_{RP}\varepsilon_{RP}cx.$$

As above, we start with the flexible price expression for related party profits: $\pi_{RP}^{A} = \frac{1}{\eta}pc = \frac{1}{\eta}p^{1-\eta}$. *Partially differentiating with respect to the distributor's unit input cost gives:* $\frac{\partial\pi_{RP}^{A}}{\partial x} = \frac{1-\eta}{\eta}p^{-\eta}\frac{\partial p}{\partial x} = -c$ *because* $\frac{\partial p}{\partial x} = \frac{\eta}{\eta-1}$. *Using:*

$$\begin{aligned}
\frac{\partial x}{\partial p_{RP}} &= \left[\gamma_{AL}^{\rho}p_{AL}^{1-\rho}+\left(1-\gamma_{AL}\right)^{\rho}{p_{RP}}^{1-\rho}\right]^{\frac{\rho}{1-\rho}}\left(\gamma_{RP}\right)^{\rho}{p_{RP}}^{-\rho} \\
&= c_{RP}/c,
\end{aligned}$$

we can write: $\frac{\partial \pi^A_{RP}}{\partial m_{RP}} = \frac{\partial \pi^A_{RP}}{\partial x}\frac{\partial x}{\partial p}\frac{\partial p}{\partial m_{RP}} = -c_{RP}$, *because* $p_{RP} = m_{RP}$ *and, hence,* $\frac{dp_{RP}}{dm_{RP}} = 1$. *The remaining steps follow those in Claim 1.*

Claim 3 *We define* $\eta/(2\eta - 1) = \overline{s_{AL}}$ *and wish to show that:*

$$s_{AL} < \overline{s_{AL}} \Longrightarrow \Omega_{RP} > \Omega_{AL}.$$

We write:

$$\begin{aligned}
\Omega_{RP} &= s_{RP}\varepsilon_{RP}cx \\
&= (1 - s_{AL})(\eta - s_{AL}(\eta - \rho))cx \\
&= \left(\eta\left(1 - 2s_{AL} + s_{AL}^2\right) + \rho s_{AL}(1 - s_{AL})\right)cx,
\end{aligned}$$

and

$$\begin{aligned}
\Omega_{AL} &= (\varepsilon_{AL} - 1)\alpha s_{AL}cx \\
&= \left(\eta s_{AL}^2 + \rho s_{AL}(1 - s_{AL}) - s_{AL}\right)\alpha cx.
\end{aligned}$$

In this form, it is easy to see:

$$(\alpha\Omega_{RP} - \Omega_{AL})/\alpha cx = \eta(1 - 2s_{AL}) + s_{AL}.$$

Factoring out the arm's length share, we see that:

$$s_{AL} < \eta/(2\eta - 1) \Longrightarrow \alpha\Omega_{RP} > \Omega_{AL},$$

and since $\alpha < 1$, *this implies* $\Omega_{RP} > \Omega_{AL}$.

Appendix B: Model Solution and Simulation

This appendix gives details of the projection method used to find an approximate solution to the model in Section 2 and to generate simulated data.[19] Application of these methods to a model of adjustment costs follows Midrigan (2010, 2011) and their use for solving a dynamic game follows Miranda and Vedenov (2001). Miranda and Fackler (2002) provides an accompanying MATLAB toolbox (CompEcon) that was used extensively.

I approximate each of the two expected value functions (7) with a linear combination of orthogonal (Chebyshev) basis polynomials:

$$V_j(\Theta_t) \approx \sum_{i_1=1}^{N_1} \sum_{i_2=1}^{N_2} \sum_{i_3=1}^{N_3} \sum_{i_4=1}^{N_4} b_{i_1 i_2 i_3 i_4} \psi_{i_1}(p_{1,t-1})\, \psi_{i_2}(p_{2,t-1})\, \psi_{i_3}(m_1)\, \psi_{i_4}(m_2) \tag{B1}$$

where ψ_{i_j} is an i_jth degree Chebyshev polynomial and is a function of the jth state variable. The collocation method requires the approximation (B1) to hold exactly at specific points called collocation nodes: $\{p_{1,t-1}(i_{n_1}), p_{2,t-1}(i_{n_2}), m_1(i_{n_3}), m_2(i_{n_4})\}$ for $i_{n_k} = 1...N_k$ and $k = 1...4$. Since there are two value functions to estimate (one for each firm), this reduces the problem to solving a system of $2N_1N_2N_3N_4$ equations in $2N_1N_2N_3N_4$ unknown coefficients, $b_{i_1 i_2 i_3 i_4}$.

The algorithm starts with a guess for the coefficients on the Chebyshev basis polynomials and the optimal policies for each firm at each collocation node. Since the approximated function is an expected (rather than realized) value function, this policy is the profit maximizing price, conditional on an adjustment cost sufficiently low to warrant a price change. This potential price (together with the distribution function $G_j(\phi)$) implicitly defines the probability of price adjustment.

Given the initial set of collocation coefficients and taking the guess for the other firm's optimal policy as given, I use a modified Newton routine to solve simultaneously for each firm's optimal price, conditional on adjustment, at each collocation node. The first order condition (FOC) has a term reflecting profits given an adjustment price as well as the expected continuation value given this price. In order to approximate this latter term, I discretize the joint distribution of cost (exchange rate) shocks and integrate using Gaussian quadrature. After each Newton step, I calculate the probability of adjustment, ξ_j, implied by the optimal adjustment price because this probability enters the competitor's own optimization problem (6). This process continues until the FOC of both firms is sufficiently close to zero and the probability of adjustment does not change with additional iterations.

Finally, a combination of function iteration with dampening and Newton's method with backstepping is used to determine the next set of Chebyshev polynomial coefficients to consider. With this new set of collocation coefficients, a new set of equilibrium policies is found. The process is repeated until the changes in the basis coefficients and optimal policies in each iteration, as well as

[19] I thank Uli Doraszelski for his very helpful advice on the numerical methods detailed in this section.

the differences between the right-hand side and left-hand side of the expected value function (7) at the collocation nodes, are extremely small.

The accuracy of the approximations can be gauged by calculating the difference between the left- and right-hand sides of the firm's expected value functions at a set of nodes denser than the collocation nodes. For some of the parameter configurations tested, these errors are larger than would be desirable, at average respective levels of about 1e-4 and 5e-4 and maximum levels of about 7e-4 and 4e-3 for the related party and arm's length firms, when expressed as a share of the expected value functions. This lack of precision, in addition to the two-firm structure, precludes treatment of the simulation as a true calibration exercise. The consistency of the comparative statics and qualitative results across approximations with varying numbers of collocation nodes, however, suggests this level of accuracy is sufficient to demonstrate the key points in this paper.[20]

The above procedure generates a solution for a given set of parameter values. To consider other parameter values, I start with the solution to a close by problem (in the sense that the parameter values are close) and use simple continuation methods. There are no guarantees these will work, however, and I often had to try varying multiple parameters, including the number of collocation nodes itself, in order to move around the parameter space.[21] Once a solution to the above system of equations has been approximated, I simulate the cost shocks and generate simulated pricing responses from the firms.

There is no way to guarantee a suitable starting guess for policies from new locations in the parameter space (after random cost shocks), so the algorithm occasionally does not converge. In such cases, I simply draw a different shock value and try again. This is a frequent occurrence for the least differentiated sectors, though is very rare in the remaining sectors. With simulated cost and pricing data, I generate measures for key statistics such as the unconditional duration (or stickiness) of prices, the synchronization of price changes, and the pass-through of cost shocks.

[20] Given the very similar results for varying numbers of nodes, most results in the figures reflect faster simulations with less nodes than that used to measure the size of approximation errors.

[21] See Chapter 5 of Judd (1988) for a discussion of simple continuation methods.